The CIA (consistency in aggregation) approach
A new economic approach to elementary indices

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* This presentation represents the author’s personal opinions and does not necessarily reflect the views of the Deutsche Bundesbank or its staff.
Outline

1. Motivation
2. Background
3. Economic approach
4. Consistency approach
5. Empirical results
6. Discussion

“Elementary, my dear Watson!” (Sherlock Holmes)
1. Motivation

- Practical **consumer price indices** are constructed in two stages:

  1. **a first stage at the lowest level of aggregation** where price information is available but associated expenditure or **quantity information** is not available and
  2. **a second stage of aggregation** where expenditure information is available at a higher level of aggregation.

- Although the scope of the discussion is on **the choice of the index formula at the elementary level**, this choice eventually depends on **the target price index** at the aggregate level.

- HICP is not intended to measure the cost of living (COLI), rather, it is a **cost of goods index (COGI)**.
2. Background
Bilateral price indices

− We specify **two accounting periods**, \( t \in \{0, 1\} \), for which we have micro price and quantity data for \( n \) commodities (bilateral index context).
− Denote the **price and quantity** of commodity \( i \in \{1, \ldots, n\} \) in period \( t \) by \( p_i^t \) and \( q_i^t \), respectively.

− A very simple approach to the determination of a price index over a group of commodities is **the (fixed) basket approach**.
− Define the **Lowe (1823) price index**, \( P_{Lo} \), as follows:

\[
P_{Lo} = \frac{\sum_{i=1}^{n} p_i^1 \cdot q_i}{\sum_{i=1}^{n} p_i^0 \cdot q_i}.
\]

− There are **two natural choices** for the reference basket:
  • the period 0 commodity vector \( q^0 = (q_1^0, \ldots, q_n^0) \) or
  • the period 1 commodity vector \( q^1 = (q_1^1, \ldots, q_n^1) \).
2. Background
Laspeyres, Paasche and Fisher

− These two choices lead to
  • the Laspeyres (1871) price index $P_L$, if we choose $q = q^0$, and
  • the Paasche (1874) price index $P_P$, if we choose $q = q^1$:

$$P_L = \frac{\sum_{i=1}^{n} p_i^1 \cdot q_i^0}{\sum_{i=1}^{n} p_i^0 \cdot q_i^0}, \quad P_P = \frac{\sum_{i=1}^{n} p_i^1 \cdot q_i^1}{\sum_{i=1}^{n} p_i^0 \cdot q_i^1}.$$

− According to the CPI Manual (ILO et al., 2004), “the Paasche and Laspeyres price indices are equally plausible.”
− Taking an evenly weighted average of these basket price indices leads to symmetric averages.
− The geometric mean, which leads to the Fisher (1922) price index, $P_F$, is defined as:

$$P_F = \sqrt{P_L \cdot P_P}.$$
2. Background
Keynes’ pure theory of money

− In his 1930 A Treatise on Money (pp. 95-120), Keynes deals with the theory of comparisons of purchasing power.
− Comparisons of purchasing power mean comparisons of the command of money over two collections of commodities which are in some sense “equivalent” to one another, and not over quantities of utility.

− Applying the “method of limits” establishes that in any case the measure of the change in the value of money lies between the Laspeyres and Paasche price indices.

− The “crossing of formulae”, to which Fisher has devoted much attention, is, in effect, an attempt to carry the method of limits somewhat further – further (in Keynes’ opinion) than is legitimate.

− We can concoct all sorts of algebraic function of $P_L$ and $P_P$, and there will not be a penny to choose between them.
2. Background
Elementary indices

- Suppose that there are \( M \) lowest-level items or specific commodities in a chosen elementary category.
- Denote the period \( t \) price of item \( m \) by \( p_m^t \) for \( t \in \{0, 1\} \) and for items \( m \in \{1, \ldots, M\} \).
- The Dutot (1738) elementary price index, \( P_D \), is equal to the arithmetic average of the \( M \) period 1 prices divided by the arithmetic average of the \( M \) period 0 prices.
- The Carli (1764) elementary price index, \( P_C \), is equal to the arithmetic average of the \( M \) item price ratios or price relatives, \( p_m^1/p_m^0 \).
- The Jevons (1865) elementary price index, \( P_J \), is equal to the geometric average of the \( M \) item price ratios or price relatives, \( p_m^1/p_m^0 \), or the geometric average of the \( M \) period 1 prices divided by the geometric average of the \( M \) period 0 prices.

\[
\begin{align*}
P_D &= \frac{1}{M} \sum_{m=1}^{M} p_m^1 \quad P_C = \frac{1}{M} \sum_{m=1}^{M} \frac{p_m^1}{p_m^0}, \\
P_J &= \sqrt[M]{\prod_{m=1}^{M} p_m^1} = \frac{1}{M} \sqrt[M]{\prod_{m=1}^{M} p_m^0}.
\end{align*}
\]
2. Background
Is the Carli index really “upward biased”?

The sole argument frequently put forward why the Carli index should be abandoned, is the claim that it has an “upward bias” with reference to the time reversal test or circularity test (cf. Diewert, 2012):

\[ P_C(p^0, p^1) \cdot P_C(p^1, p^2) = P_C(p^0, p^1) \cdot P_C(p^1, p^0) \geq 1 = P_C(p^0, p^0) \text{ for } p^2 = p^0. \]

But this argument is useless in the bilateral index context where we can compare the two periods under consideration directly, i.e. there is no bias at all:

\[ P_C(p^0, p^2) = P_C(p^0, p^0) = 1 \text{ for } p^2 = p^0. \]

In the context of chain indices, the elementary aggregates only feed into the higher-level indices in which the elementary price indices – comparing periods \( t-1 \) and \( t (!) \) – are averaged using a set of pre-determined weights (chain indices are non-aggregable); the Dutot, Carli and Jevons indices are, thus, not chain-linked.

What is more, it apparently fell into oblivion that the then chain-linked Laspeyres, Paasche, Fisher, Walsh and Törnqvist price indices are subject to chain drift; i.e. all chain indices are path dependent, which is the opposite of transitivity.
3. Economic approach

- The CPI Manual, paragraphs 20.71-20.86, has a section in it which describes an economic approach to elementary indices.

- This section has sometimes been used to justify the use of the Jevons index, i.e. the geometric mean, over the use of the Carli index, i.e. the arithmetic mean, or vice versa depending on how much substitutability exists between items within an elementary stratum.

- This is a misinterpretation of the analysis that is presented in this section of the Manual.

- Thus, the economic approach cannot be applied at the elementary level unless price and quantity information are both available.

- Such information is typically not available, which is exactly the reason elementary indices are used rather than target indices. (Diewert, 2012, “Consumer Price Statistics in the UK”)
4. Consistency approach
Consistency in aggregation

- The consistency in aggregation (CIA) approach newly developed (Mehrhoff, 2010, Jahr. Nationalökon. Statist.) fills the void of guiding the choice of the elementary index (for which weights are not available) that corresponds to the characteristics of the index at the second stage (where weights are actually available).

- It contributes to the literature by looking at how numerical equivalence between an unweighted elementary index and a weighted aggregate index can be achieved, independent of the axiomatic properties.

- Consistency in aggregation means that if an index is calculated stepwise by aggregating lower-level indices to obtain indices at progressively higher levels of aggregation, the same overall result should be obtained as if the calculation had been made in one step.
4. Consistency approach
Cost of goods index

− To reiterate, **we measure the change in the cost of purchasing a fixed basket of goods and services**, and *not* the change in the minimum cost of maintaining a given level of utility or welfare.

− The use of the Dutot and Carli formulae at the elementary level of aggregation for *homogeneous* items can be perfectly **consistent with a Laspeyres index concept**.

− The **Laspeyres price index** can be rewritten in an alternative manner as follows:

\[
P_L = \frac{\sum_{m=1}^{M} p_m^1 \cdot q_m^0}{\sum_{m=1}^{M} p_m^0 \cdot q_m^0} = \sum_{m=1}^{M} p_m^1 \cdot \frac{p_m^0 \cdot q_m^0}{\sum_{l=1}^{M} p_l^0 \cdot q_l^0} = \sum_{m=1}^{M} \frac{p_m^1}{p_m^0} \cdot s_m^0,
\]

− where \( s_m^0 \) is the **period 0 expenditure share** on commodity \( m \).
4. Consistency approach
A thought experiment

− The first case is where the underlying preferences are Leontief preferences, i.e. consumers prefer not to make any substitutions in response to changes in relative prices (zero elasticity):

\[ q_m^0 = q_m^1 = q \text{ and, hence, } P_L = \frac{\sum_{m=1}^{M} p_m^1 \cdot q}{\sum_{m=1}^{M} p_m^0 \cdot q} = \frac{1}{M} \sum_{m=1}^{M} p_m^1 = P_D. \]

− The second case is when the preferences can be represented by a Cobb-Douglas function, i.e. consumers vary the quantities in inverse proportion to the changes in relative prices so that expenditure shares remain constant (unity elasticity):

\[ s_i^0 = s_i^1 = M^{-1} \text{ and, hence, } P_L = \sum_{m=1}^{M} \frac{p_m^1}{p_m^0} \cdot M^{-1} = \frac{1}{M} \sum_{m=1}^{M} \frac{p_m^1}{p_m^0} = P_C. \]
4. Consistency approach
Generalised means

− A single comprehensive framework, known as generalised means, unifies the aggregate and elementary levels.

− The generalised mean of order $r$ for the $M$ item price ratios or price relatives, $p_m^1/p_m^0$, is defined as follows:

$$P^r = \begin{cases} 
\sqrt[r]{\frac{1}{M} \sum_{m=1}^{M} \left( \frac{p_m^1}{p_m^0} \right)^r} & \text{if } r \neq 0, \\
\prod_{m=1}^{M} \frac{p_m^1}{p_m^0} & \text{if } r = 0.
\end{cases}$$

− The generalised mean represents a whole class of unweighted elementary indices, such as the Carli and Jevons indices for $r = 1$ and $r = 0$, respectively.
4. Consistency approach
Typical shape

\[ P_{\min} = \min \left( \frac{p_{m}^i}{p_{m}^0} \right), \quad P_{\max} = \max \left( \frac{p_{m}^i}{p_{m}^0} \right) \]
4. Consistency approach
CES aggregator function

− It is supposed that the unit cost function has the following functional form:

\[
\begin{cases}
\alpha_0 \cdot \left( \sum_{m=1}^{M} \alpha_m \cdot p_m^{1-\sigma} \right)^{1/(1-\sigma)} & \text{if } \sigma \neq 1, \\
\alpha_0 \cdot \prod_{m=1}^{M} p_m^{\alpha_m} & \text{if } \sigma = 1,
\end{cases}
\]

− where the \( \alpha_m \) are non-negative consumer preference parameters with \( \sum_{m=1}^{M} \alpha_m = 1 \).
− This unit cost function corresponds to a CES aggregator or utility function.
− The parameter \( \sigma \) is the elasticity of substitution:
  • When \( \sigma = 0 \), the underlying preferences are Leontief preferences.
  • When \( \sigma = 1 \), the corresponding utility function is a Cobb-Douglas function.
4. Consistency approach
Laspeyres and Paasche price indices

− A generalised mean of order $r$ equal to the elasticity of substitution ($\sigma$) yields approximately the same result as the Laspeyres price index.

− Hence, if the elasticity of substitution is one (Cobb-Douglas preferences), for example, $r$ must equal one and the Carli index at the elementary level will correspond to the Laspeyres price index as target index.

− However, if the Paasche price index should be replicated, the order of the generalised mean must equal minus the elasticity of substitution, in the above example minus one.

− Thus, the harmonic index gives the same result and therefore, in this case it should be used at the elementary level.

− Only if the elasticity of substitution is zero (Leontief preferences), the Jevons (Dutot) index corresponds to both the Laspeyres and Paasche price indices – which in this case coincide.
5. Empirical results
Alcoholic beverages in the UK

− As an empirical application, detailed expenditure data from Kantar Worldpanel for elementary aggregates within the COICOP group of alcoholic beverages in the UK are analysed.

− The data cover the period from January 2003 to December 2011; the data set consists of transaction level data, which records inter alia purchase price and quantity, and includes 192,948 observations after outlier identification.

− The elasticity of substitution is estimated in the framework of a log-linear model by means of ordinary least squares. (Note that the consumer preference parameters are removed via differencing products common to adjacent months and, thus, there is no need for application of seemingly unrelated regression.)

− As a robustness check to the CES model based results, the generalised mean which minimises relative bias and root mean squared relative error to the desired aggregate index is found directly by numerical optimisation techniques. (Rather than at the aggregate transaction level, like the econometric method, this analysis, however, is performed one level above – at the elementary level.)
5. Empirical results

COICOP structure for alcoholic beverages
5. Empirical results
Findings on substitution behaviour

- The median elasticity of substitution is 1.5, ranging from .7 to 3.8.
- All estimates are statistically significantly greater than zero; for 8 out of 19 sub-classes the difference to iso-elasticity is insignificant, while for the remaining 11 sub-classes substitution is found to even exceed unity elasticity.
- In spirits, consumers are more willing to substitute between different types of whiskey (S 4) than is the case for brandy or vodka (S 1 and S 2).
- For both red and white wines, substitution is more pronounced for the New World (W 5 and W 7) than for European wines (W 4 and W 6).
- Also, the elasticity of substitution tends to be higher for 12 cans and 20 bottles of lager (B 3 and B 5), respectively, than for 4 packs (B 2 and B 4).
- These results are consistent with the findings of Elliott and O’Neill (2012).
- Furthermore, comparing the CES regression results with the direct calculation of the generalised means, the outcomes do not change qualitatively.
- In particular, the Carli index performs remarkably well at the elementary level of a Laspeyres price index, questioning the argument of its “upward bias” – in fact, it is the Jevons index that has a downward bias.
5. Empirical results

CES estimation results

[Graph showing CES estimates with 95% confidence interval]
5. Empirical results
Laspeyres price index: robustness

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CES estimates robustness check – Laspeyres price index

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5. Empirical results
Laspeyres price index: bias

Relative biases of elementary indices – Laspeyres price index

in %

S 1  S 2  S 3  S 4  W 1  W 2  W 3  W 4  W 5  W 6  W 7  W 8  W 9  B 1  B 2  B 3  B 4  B 5  B 6

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5. Empirical results
Laspeyres price index: time series

Laspeyres price index for W 5 – Red wine (New World)

Previous month = 100, log scale

Carli index \( (r = 1) \) vs. CES estimate \( (r = 3) \)

Jevons index \( (r = 0) \) vs. CES estimate \( (r = 3) \)
6. Discussion

- The problem of aggregational consistency demonstrates the need for a weighting at the lowest possible level.
- This would mean that, in the trade-off between estimated weights/weights from secondary sources on the one hand and the elementary bias of unweighted indices on the other, the balance would often tip in favour of weighting.
- The biases at the elementary level can, in some cases, reach such large dimensions that they become relevant for the aggregate index.
- There is a “price” to be paid at the upper level for suboptimal index formula selection at the lower level; thus, the need for two-staged price indices to be accurately constructed becomes obvious.
- Disaggregation is a panacea!

- Insofar as no information on weights is available, studies on substitution can help in guiding the choice of the optimal elementary index for a given measurement target.
- Often, even an expert judgement on substitutability outperforms the test approach.